

Studying conformally flat spacetimes with an elastic stress energy tensor using 1+3 formalism

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Abstract.

Conformally flat spacetimes with an elastic stress energy tensor given by a diagonal trace-free anisotropic pressure tensor are investigated using 1+3 formalism. We show how the null tetrad Ricci components are related to the pressure components and energy density. The 1+3 Bianchi and Jacobi identities and Einstein field equations are written for this particular case. In general the commutators must be considered since they supply potentially new information on higher order derivatives of the 1+3 quantities. We solve the system for the non rotating case which consist of ODEs of a spatial coordinate.

1 Introduction

The theory of elasticity in the context of general relativity was developed in the mid twentieth century. The need for such a theory came in the late 1950s with Webers bar antenna for gravitational waves [1], in order to explain how these waves interact with elastic solids. Actually, for this phenomena the weak-field approximation was sufficient in the treatment of the problem given by Weber. Only in 1973, in a paper by Carter and Quintana [2], did a fully developed non-linear theory of elasticity adapted to general relativity appear, remaining to this day a standard reference in the field, although the basic theoretical framework of this theory had already been given by Souriau in [3]. Also before the article by Carter and Quintana, work by Maugin made considerable contributions to the field [4]-[5]. Lately, the theory of elasticity in general relativity was reconsidered by Magli and Kijowski [6]-[7] and Chritodoulou [8], in this work they explore the gauge character of relativity. Authors such as Beig and Schmidt, have proven several existence and uniqueness theorems [9]. More recently, Karlovini

and Samuelsson have given a self contained formulation of general relativistic elasticity in [10]. They applied the theory of elasticity to spherically symmetric space-times and studied radial and axial perturbations [11]-[13]. Park [14] established existence theorems for spherically symmetric static solutions for elastic bodies and Brito, Varot and Vaz [15] have obtained static shear-free and non-static shear-free solutions for spherically symmetric elastic spacetimes. Calogero and Heinzle [16] studied the dynamics of Bianchi type I elastic spacetimes.

In this work we study a very simple elastic model with a diagonal trace-free anisotropy pressure tensor using the 1+3 extended frame approach and following the notation given in Uggla [17]. We note that work on the covariant 1+3 splitting of fluid spacetime geometries was first initiated by Eisenhart and Synge and continued by Gödel, Raychaudhuri and other authors such as Schücking, Ehlers, Sachs and Trümper (related references [18], [19], [20], [21]). In the paper by Uggla the basic dynamical equations of the extended 1+3 orthonormal frame approach are explicitly given in terms of variables that are naturally adapted to the 1+3 structure, and they include the Bianchi and Jacobi identities, the Einstein field equations and the commutators. This formulation is analogous to the Newmann Penrose approach [22] in the sense that a null congruence is replaced by a timelike congruence. The general properties of the 1+3 orthonormal frame can be found in books such as Wald [23] and Felice and Clarke [24] and in Edgar [25]. Several applications have been discussed in Pirani [26], Ellis [27] and MacCallum [28]. A more complete list of references can be found in [17].

The organization of this paper is as follows. In section 2 we outline the theory of the 1+3 formalism and present the 1+3 split of the commutators, curvature variables and their field equations, namely Bianchi and Jacobi identities and Einstein field equations. We will follow the same notation convention for tensor indices used in [17]. Spacetime coordinate tensor indices will be denoted by letters from the second half of the Greek alphabet ($\mu, \nu, \rho, \dots = 0 - 3$) while spatial coordinate indices are represented by letters from the second half of the Latin alphabet ($i, j, k, \dots = 1, \dots, 3$). Orthonormal frame indices will be denoted by letters from the first half of the Latin alphabet ($a, b, c, \dots = 0, \dots, 3$) while spatial frame indices are chosen from the first half of the Greek alphabet ($\alpha, \beta, \gamma, \dots = 1 - 3$). In section 3 we relate the Ricci spinor components to the 1+3 quantities. In section 4 we study conformally flat spacetimes with an elastic source given by a diagonal trace-free anisotropy pressure tensor using the 1+3 formalism. We determine the 1+3 equations for a very simple case of non rotating solutions. In particular we study the non rotating case where the system obtained is an ODE system of a spatial coordinate.

2 1+3 Formalism

When studying a dynamical model in general relativity possessing an energy-momentum-stress tensor with a timelike eigendirection, for example perfect fluids, the associated timelike vector field u on the spacetime (M, g) determines

the projection tensors U and h , which project parallel and orthogonal to u in the tangent space at each point $p \in (M, g)$, respectively. u is chosen to be a unit timelike vector:

$$u_\mu u^\mu = -1, \quad (1)$$

and the projection tensors U and h are defined by

$$U^\mu{}_\nu = -u^\mu u_\nu \quad (2)$$

$$h^\mu{}_\nu = \delta^\mu{}_\nu + u^\mu u_\nu. \quad (3)$$

Due to the existence of this singled out timelike direction u a covariant 1+3 tensor decomposition of all geometrical objects of physical value can be made using U and h .

We will denote covariant derivative by ∇ and the totally antisymmetric permutation tensor by $\eta^{\mu\nu\rho\sigma}$.

The well known kinematical fields associated with the timelike congruence u are defined by

$$\dot{u}^\mu = h^\mu{}_\nu u^\rho \nabla_\rho u^\nu \quad (4)$$

$$\Theta = h^\mu{}_\nu \nabla_\mu u^\nu \quad (5)$$

$$\sigma_{\mu\nu} = [h^\rho{}_{(\mu} h^\sigma{}_{\nu)} - \frac{1}{3} h_{\mu\nu} h^{\rho\sigma}] (\nabla_\rho u_\sigma) \quad (6)$$

$$\omega_{\mu\nu} = -h^\rho{}_{[\mu} h^\sigma{}_{\nu]} (\nabla_\rho u_\sigma), \quad (7)$$

where \dot{u}^μ is the acceleration vector, Θ the rate of expansion scalar, $\sigma_{\mu\nu}$ is the rate of shear tensor and $\omega_{\mu\nu}$ is the vorticity tensor. The magnitude of the rate of shear σ , the vorticity vector ω^μ and the magnitude of the vorticity ω are defined as

$$\sigma^2 = \frac{1}{2} \sigma_{\mu\nu} \sigma^{\mu\nu} \quad (8)$$

$$\omega^\mu = \frac{1}{2} \eta^{\mu\nu\rho\sigma} \omega_{\nu\rho} u_\sigma \quad (9)$$

$$\omega^2 = \frac{1}{2} \omega_{\mu\nu} \omega^{\mu\nu} = \omega_\mu \omega^\mu. \quad (10)$$

The vector field u is hypersurface forming if $\omega = 0$.

In the orthonormal frame approach one chooses at each point of the space-time manifold (M, g) a set of four linearly independent 1-forms e^a such that the line element is given by

$$ds^2 = \eta_{ab} e^a e^b, \quad (11)$$

where $\eta_{ab} = \text{diag}[-1, 1, 1, 1]$ is the constant Minkowskian frame metric. The vectors e_a represent the dual basis.

In the 1+3 orthonormal frame formalism one aligns the timelike direction of the orthonormal frame with the tangent of the preferred timelike congruence ($e_0 = u$).

2.1 Commutators

The commutator functions are defined by

$$[e_a, e_b] = \gamma^c_{ab} e_c, \quad (12)$$

where the frame vectors e_a act as differential operators on geometrical objects. The Ricci rotation coefficients Γ^a_{bc} can be written in terms of the commutation functions as

$$\gamma^a_{bc} = \Gamma^a_{cb} - \Gamma^a_{bc}. \quad (13)$$

The commutator functions with one or two indices equal to zero can be expressed in terms of the frame components of the kinematic quantities associated with the timelike congruence. The angular velocity Ω^a is defined by

$$\Omega^a = \frac{1}{2} \eta^{abcd} e_b \cdot \nabla_u (e_c) u_d. \quad (14)$$

The purely spatial components of the commutation functions $\gamma^\alpha_{\beta\gamma}$ can be decomposed [29] a_α and $n_{\alpha\beta}$ by means of

$$\gamma^\alpha_{\beta\gamma} = 2a_{[\beta} \delta^\alpha_{\gamma]} + \epsilon_{\beta\gamma\delta} n^{\delta\alpha}, \quad (15)$$

where $n_{\alpha\beta}$ is symmetric and $\epsilon_{\beta\gamma}$ is the totally antisymmetric three-dimensional permutation tensor.

The commutators are given by expressions for γ^a_{bc} and their 1+3 decomposition results in the following equations

$$[e_0, e_\alpha] = \dot{u}_\alpha e_0 - \left[\frac{1}{3} \Theta \delta^\beta_\alpha + \sigma^\beta_\alpha + \epsilon^\beta_{\alpha\gamma} (\omega^\gamma - \Omega^\gamma) \right] e_\beta \quad (16)$$

$$[e_\alpha, e_\beta] = -2\epsilon_{\alpha\beta\gamma} \omega^\gamma e_0 + [2a_{[\alpha} \delta^\gamma_{\beta]} + \epsilon_{\alpha\beta\delta} n^{\delta\gamma}] e_\gamma. \quad (17)$$

2.2 Curvature variables and their field equations

The relation between the Riemann curvature tensor and the Ricci rotation coefficients is given by

$$R^a_{bcd} = e_c(\Gamma^a_{bd}) - e_d(\Gamma^a_{bc}) + \Gamma^a_{ec} \Gamma^e_{bd} - \Gamma^a_{ed} \Gamma^e_{bc} - \Gamma^a_{be} \Gamma^e_{cd}, \quad (18)$$

and the 16 Jacobi identities are then given by the following equations

$$R^a{}_{[bcd]} = 0; \quad R_{abcd} = R_{cdab}. \quad (19)$$

The Einstein field equations with a non zero cosmological constant can be written as

$$R_{ab} = T_{ab} - \frac{1}{2}Tg_{ab} + \Lambda g_{ab}, \quad (20)$$

and the 1+3 decomposition of the energy-momentum-stress tensor T_{ab} with respect to the timelike vector field u in the fluid description of phenomena of a matter source is given by

$$T_{ab} = \mu u_a u_b + 2q_{(a} u_{b)} + p h_{ab} + \pi_{ab}, \quad (21)$$

where μ denotes the total energy density scalar, p denotes the isotropic pressure scalar, q^a denotes the energy current density vector, and π_{ab} denotes the anisotropic pressure tensor. Note that

$$q_a u^a = 0; \quad \pi_{ab} u^b = 0; \quad \pi^a{}_a = 0; \quad \pi_{ab} = \pi_{ba}. \quad (22)$$

The matter fields need to satisfy an appropriate thermodynamical equation of state in order to describe the physics of the underlying fluid spacetime geometry under consideration.

The completely tracefree Weyl conformal curvature tensor is defined by

$$C^{ab}{}_{cd} = R^{ab}{}_{cd} - 2\delta^{[a}{}_{[c} R^{b]}{}_{d]} + \frac{1}{3}R\delta^a{}_{[c}\delta^b{}_{d]}. \quad (23)$$

When there is a singled out timelike direction u , it is convenient to decompose the Weyl tensor into its electric part given by

$$E_{ab} = C_{cedf} h^c{}_a u^e h^d{}_b u^f, \quad (24)$$

and magnetic part

$$H_{ab} = {}^*C_{cedf} h^c{}_a u^e h^d{}_b u^f, \quad (25)$$

where ${}^*C_{abcd}$ is the dual of C_{abcd} defined by

$${}^*C_{abcd} = \frac{1}{2}\eta_{ab}{}^{ef}C_{efcd}. \quad (26)$$

The electric and magnetic tensors are symmetric and tracefree and satisfy $E_{ab}u^b = H_{ab}u^b = 0$ and lead to the expression

$$\begin{aligned} C^{ab}{}_{cd} &= [4\delta^{[a}{}_{[e}\delta^{b]}{}_{f]}\delta^g{}_{[c}\delta^h{}_{d]} - \eta^{ab}{}_{ef}\eta^{gh}{}_{cd}]E^e{}_g u^f u_h \\ &- 2[\eta^{ab}{}_{ef}\delta^g{}_{[c}\delta^h{}_{d]} + \delta^{[a}{}_{[e}\delta^{b]}{}_{f]}\eta^{gh}{}_{cd}]H^e{}_g u^f u_h. \end{aligned} \quad (27)$$

Considering equation (23) and using (20), (21) and (27) we obtain the following expression for the Riemann curvature tensor

$$\begin{aligned}
R^{ab}{}_{cd} &= [4\delta^{[a}{}_e\delta^{b]}{}_f\delta^g{}_{[c}\delta^h{}_{d]} - \eta^{ab}{}_{ef}\eta^{gh}{}_{cd}]E^e{}_g u^f u_h \\
&- 2[\eta^{ab}{}_{ef}\delta^g{}_{[c}\delta^h{}_{d]} + \delta^{[a}{}_e\delta^{b]}{}_f\eta^{gh}{}_{cd}]H^e{}_g u^f u_h \\
&+ 2\delta^{[a}{}_c[(\mu + p)u^b]u_d + q^b]u_d + u^b]q_d + \pi^b]_d] \\
&+ \frac{2}{3}(\mu + \Lambda)\delta^a{}_{[c}\delta^b{}_{d]}.
\end{aligned} \tag{28}$$

Inserting expression (18) for the Riemann tensor into the left hand side of (28) one obtains the 10 Einstein field equations, the 16 Jacobi identities and expressions for E_{ab} and H_{ab} in terms of the basis variables

$$\{\Theta, \dot{u}_\alpha, \sigma_{\alpha\beta}, \omega_\alpha, \Omega_\alpha, a_\alpha, n_{\alpha\beta}, \mu, p, q_\alpha, \pi_{\alpha\beta}\},$$

and their e_0 and e_α frame derivatives

Einstein field equations

$$\begin{aligned}
e_0(\Theta) &= -\frac{1}{3}\Theta^2 + (e_\alpha + \dot{u}_\alpha - 2a_\alpha)(\dot{u}^\alpha) - 2\sigma^2 + 2\omega^2 \\
&- \frac{1}{2}(\mu + 3p) + \Lambda
\end{aligned} \tag{29}$$

$$\begin{aligned}
e_0(\sigma^{\alpha\beta}) &= -\Theta\sigma^{\alpha\beta} + (\delta^{\gamma(\alpha}e_\gamma + \dot{u}^{(\alpha} + a^{(\alpha})(\dot{u}^{\beta)}) + 2\omega^{(\alpha}\Omega^{\beta)} + \pi^{\alpha\beta} \\
&- {}^*S^{\alpha\beta} - \frac{1}{3}\delta^{\alpha\beta}[(e_\gamma + \dot{u}_\gamma + a_\gamma)(\dot{u}^\gamma) + 2\omega_\gamma\Omega^\gamma] \\
&+ \epsilon^{\gamma\delta(\alpha}[2\Omega_\gamma\sigma^{\beta)}_\delta - n^{\beta)}_\delta\dot{u}_\gamma]
\end{aligned} \tag{30}$$

$$0 = \mu - \frac{1}{3}\Theta^2 + \sigma^2 - \omega^2 - 2\omega_\alpha\Omega^\alpha - \frac{1}{2}{}^*R + \Lambda \tag{31}$$

$$\begin{aligned}
0 &= (e_\beta - 3a_\beta)(\sigma^{\alpha\beta}) - \frac{2}{3}\delta^{\alpha\beta}e_\beta(\Theta) + n^\alpha{}_\beta\omega^\beta + q^\alpha \\
&- \epsilon^{\alpha\beta\gamma}[(e_\beta + 2\dot{u}_\beta - a_\beta)(\omega_\gamma) + n_{\beta\delta}\sigma^\delta{}_\gamma],
\end{aligned} \tag{32}$$

where

$$\begin{aligned}
{}^*S_{\alpha\beta} &= e_{(\alpha}(a_{\beta)}) + b_{\alpha\beta} - \frac{1}{3}\delta_{\alpha\beta}[e_\gamma(a^\gamma) + b^\gamma{}_\gamma] \\
&- \epsilon^{\gamma\delta}{}_{(\alpha}(e_{|\gamma|} - 2a_{|\gamma|})(n_{\beta)\delta})
\end{aligned} \tag{33}$$

$${}^*R = 2(2e_\alpha - 3a_\alpha)(a^\alpha) - \frac{1}{2}b^\alpha{}_\alpha \tag{34}$$

$$b_{\alpha\beta} = 2n_{\alpha\gamma}n^\gamma{}_\beta - n^\gamma{}_\gamma n_{\alpha\beta}. \tag{35}$$

Jacobi identities

$$\begin{aligned}
e_0(a^\alpha) &= -\frac{1}{3}(\delta^{\alpha\beta}e_\beta + \dot{u}^\alpha + a^\alpha)(\Theta) + \frac{1}{2}(e_\beta + \dot{u}_\beta - 2a_\beta)(\sigma^{\alpha\beta}) \\
&\quad - \frac{1}{2}\epsilon^{\alpha\beta\gamma}(e_\beta + \dot{u}_\beta - 2a_\beta)(\omega_\gamma - \Omega_\gamma)
\end{aligned} \tag{36}$$

$$\begin{aligned}
e_0(n^{\alpha\beta}) &= -\frac{1}{3}\Theta n^{\alpha\beta} - (\delta^{\gamma(\alpha}e_\gamma + \dot{u}^{(\alpha})(\omega^{\beta)} - \Omega^{\beta)}) + 2\sigma^{(\alpha}_\gamma n^{\beta)\gamma} \\
&\quad + \delta^{\alpha\beta}(e_\gamma + \dot{u}_\gamma)(\omega^\gamma - \Omega^\gamma) - \epsilon^{\gamma\delta(\alpha}[(e_\gamma + \dot{u}_\gamma)(\sigma^{\beta)}_\delta) \\
&\quad - 2n^{\beta)}_\gamma(\omega_\delta - \Omega_\delta)]
\end{aligned} \tag{37}$$

$$\begin{aligned}
e_0(\omega^\alpha) &= -\frac{2}{3}\Theta\omega^\alpha + \sigma^\alpha_\beta\omega^\beta + \frac{1}{2}n^\alpha_\beta\dot{u}^\beta - \epsilon^{\alpha\beta\gamma}[\frac{1}{2}(e_\beta - a_\beta)(\dot{u}_\gamma) \\
&\quad + \omega_\beta\Omega_\gamma]
\end{aligned} \tag{38}$$

$$\begin{aligned}
0 &= (e_\beta - 2a_\beta)(n^{\alpha\beta}) - \frac{2}{3}\Theta\omega^\alpha - 2\sigma^\alpha_\beta\omega^\beta + \epsilon^{\alpha\beta\gamma}[e_\beta(a_\gamma) \\
&\quad + 2\omega_\beta\Omega_\gamma]
\end{aligned} \tag{39}$$

$$0 = (e_\alpha - \dot{u}_\alpha - 2a_\alpha)(\omega^\alpha). \tag{40}$$

The Bianchi identities are given by

$$\nabla_{[a}R_{bc]de} = 0 \iff -\nabla_{[a}[R^c_{b}] - \frac{1}{6}\delta^c_{b}]R] = -\nabla_{[a}[T^c_{b}] - \frac{1}{3}\delta^c_{b}]T], \tag{41}$$

and can be expressed in terms of E_{ab} and H_{ab} and the 1+3 basis variables associated with T_{ab}

Bianchi identities

$$e_0(\mu) = -(\mu + p)\Theta - (e_\alpha + 2\dot{u}_\alpha - 2a_\alpha)(q^\alpha) - \sigma_{\alpha\beta}\pi^{\alpha\beta} \tag{42}$$

$$\begin{aligned}
e_0(q^\alpha) &= -\frac{4}{3}\Theta q^\alpha - \delta^{\alpha\beta}e_\beta(p) - (\mu + p)\dot{u}^\alpha \\
&\quad - (e_\beta + \dot{u}_\beta - 3a_\beta)(\pi^{\alpha\beta}) - \sigma^\alpha_\beta q^\beta + \epsilon^{\alpha\beta\gamma}[(\omega_\beta \\
&\quad + \Omega_\beta)q_\gamma + n_{\beta\delta}\pi^\delta_\gamma]
\end{aligned} \tag{43}$$

$$\begin{aligned}
e_0(E^{\alpha\beta} + \frac{1}{2}\pi^{\alpha\beta}) &= -\frac{1}{2}(\mu + p)\sigma^{\alpha\beta} - \Theta(E^{\alpha\beta} + \frac{1}{6}\pi^{\alpha\beta}) - \frac{1}{2}(\delta^{\gamma(\alpha}e_\gamma \\
&\quad + 2\dot{u}^{(\alpha} + a^{(\alpha})(q^{\beta)}) + 3\sigma^{(\alpha}_\gamma(E^{\beta)\gamma} - \frac{1}{6}\pi^{\beta)\gamma}) \\
&\quad + \frac{1}{2}n^\gamma_\gamma H^{\alpha\beta} + \frac{1}{3}\delta^{\alpha\beta}[\frac{1}{2}(e_\gamma + 2\dot{u}_\gamma + a_\gamma)(q^\gamma) \\
&\quad - 3\sigma_{\gamma\delta}(E^{\gamma\delta} - \frac{1}{6}\pi^{\gamma\delta}) + 3n_{\gamma\delta}H^{\gamma\delta}] + \epsilon^{\gamma\delta(\alpha} \\
&\quad [(e_\gamma + 2\dot{u}_\gamma - a_\gamma)(H^{\beta)}_\delta) - (\omega_\gamma - 2\Omega_\gamma)(E^{\beta)}_\delta) \\
&\quad + \frac{1}{2}\pi^{\beta)}_\delta) + \frac{1}{2}n^{\beta)}_\gamma q_\delta] - 3n^{(\alpha}_\gamma H^{\beta)\gamma}
\end{aligned} \tag{44}$$

$$\begin{aligned}
e_0(H^{\alpha\beta}) &= -\Theta H^{\alpha\beta} + 3\sigma^{(\alpha}{}_{\gamma} H^{\beta)\gamma} - \frac{3}{2}\omega^{(\alpha} q^{\beta)} - \frac{1}{2}n^{\gamma}{}_{\gamma}(E^{\alpha\beta} \\
&- \frac{1}{2}\pi^{\alpha\beta}) + 3n^{(\alpha}{}_{\gamma}(E^{\beta)\gamma} - \frac{1}{2}\pi^{\beta)\gamma}) - \delta^{\alpha\beta}[\sigma_{\gamma\delta}H^{\gamma\delta} \\
&- \frac{1}{2}\omega_{\gamma}q^{\gamma} + n_{\gamma\delta}(E^{\gamma\delta} - \frac{1}{2}\pi^{\gamma\delta})] - \epsilon^{\gamma\delta(\alpha}[(e_{\gamma} - a_{\gamma}) \\
&(E^{\beta)}_{\delta} - \frac{1}{2}\pi^{\beta)}_{\delta}) + 2\dot{u}_{\gamma}E^{\beta)}_{\delta} - \frac{1}{2}\sigma^{\beta)}_{\gamma}q_{\delta} \\
&+ (\omega_{\gamma} - 2\Omega_{\gamma})H^{\beta)}_{\delta}] \tag{45}
\end{aligned}$$

$$\begin{aligned}
0 &= (e_{\beta} - 3a_{\beta})(E^{\alpha\beta} + \frac{1}{2}\pi^{\alpha\beta}) - \frac{1}{3}\delta^{\alpha\beta}e_{\beta}(\mu) + \frac{1}{3}\Theta q^{\alpha} \\
&- \frac{1}{2}\sigma^{\alpha}{}_{\beta}q^{\beta} + 3\omega_{\beta}H^{\alpha\beta} - \epsilon^{\alpha\beta\gamma}[\sigma_{\beta\delta}H^{\delta}{}_{\gamma} + \frac{3}{2}\omega_{\beta}q_{\gamma} \\
&+ n_{\beta\delta}(E^{\delta}{}_{\gamma} + \frac{1}{2}\pi^{\delta}{}_{\gamma})] \tag{46}
\end{aligned}$$

$$\begin{aligned}
0 &= (e_{\beta} - 3a_{\beta})(H^{\alpha\beta}) - (\mu + p)\omega^{\alpha} - 3\omega_{\beta}(E^{\alpha\beta} \\
&- \frac{1}{6}\pi^{\alpha\beta}) - \frac{1}{2}n^{\alpha}{}_{\beta}q^{\beta} + \epsilon^{\alpha\beta\gamma}[\frac{1}{2}(e_{\beta} - a_{\beta})(q_{\gamma}) \\
&+ \sigma_{\beta\delta}(E^{\delta}{}_{\gamma} + \frac{1}{2}\pi^{\delta}{}_{\gamma}) - n_{\beta\delta}H^{\delta}{}_{\gamma}]. \tag{47}
\end{aligned}$$

3 Relation between the covariant 1+3 and null tetrad decomposition of the curvature tensor

3.1 Relativity

The formulation of the theory of relativistic elasticity is based on a configuration mapping $\psi : M \rightarrow X$ from the spacetime M to the three-dimensional material space X , which represents the collection of particles of the material and is equipped with a material Riemannian metric γ . The material coordinates will be denoted by y^A , $A = 1, 2, 3$, and let x^{μ} represent the coordinates in M . This mapping describes the configuration of the material and gives rise to a rank three matrix $(y^A_{\mu})_p$, $p \in M$, $y^A_{\mu} = \frac{\partial y^A}{\partial x^{\mu}}$, which is called the relativistic deformation gradient. The velocity field of matter u^{μ} a future oriented, timelike unit vector field, which spans the one-dimensional Kernel of the relativistic deformation gradient is defined by the conditions $y^A_{\mu}u^{\mu} = 0$, $u^{\mu}u_{\mu} = -1$. The pulled-back material metric

$$k_{\mu\nu} = \psi^* \gamma_{AB} = y^A_{\mu}y^B_{\nu}\gamma_{AB}, \tag{48}$$

can be used to construct the relativistic strain tensor

$$s_{\mu\nu} = \frac{1}{2}(h_{\mu\nu} - k_{\mu\nu}), \tag{49}$$

where $h_{\mu\nu} = g_{\mu\nu} + u_{\mu}u_{\nu}$, which measures the state of strain of the material. If $s_{\mu\nu} = 0$, the material is said to be in an unstrained state.

k^μ_ν has three positive eigenvalues n_1^2, n_2^2, n_3^2 . The particle density of the material is defined by $n = \sqrt{k^\mu_\nu} = n_1 n_2 n_3$ and n_1, n_2, n_3 are called linear particle densities. The energy density can be defined by $\mu = n\epsilon$, where ϵ represents the energy per particle. A constitutive equation for the material is given by specifying μ as function of three scalar invariants of k^μ_ν , $\mu = \mu(I_1, I_2, I_3)$, if we assume that the internal energy of an elastic deformation accumulated in an infinitesimal portion of the material is invariant with respect to the space-time orientation of the material. For the scalar invariants one can choose $I_1 = k^\mu_\mu$, $I_2 = k^\mu_\nu k^\nu_\mu$, $I_3 = k^\mu_\nu k^\nu_\gamma k^\gamma_\mu$. Since $n^2 = \frac{1}{6}(I_1^3 - 3I_1 I_2 + 2I_3)$, one can use the particle density n as one of the scalar invariants.

The stress-energy tensor for elastic matter is given by

$$T_{\mu\nu} = \mu u_\mu u_\nu + p_{\mu\nu}, \quad (50)$$

where the pressure tensor $p_{\mu\nu}$ is defined by

$$p_{\mu\nu} = p h_{\mu\nu} + \pi_{\mu\nu}, \quad (51)$$

p being the isotropic pressure and π_{ab} the anisotropic pressure tensor.

An equation of state that is compatible with the stress energy tensor (50) can be written as

$$\mu = \tilde{\mu} + \tilde{\rho}\sigma^2, \quad (52)$$

where $\tilde{\mu}$ is the unsheared energy density $\tilde{\rho}$ is the modulus of rigidity, σ^2 the shearing scalar and

$$p = \tilde{p} + (\tilde{\Omega} - 1)\sigma, \quad \tilde{p} = n^2 \frac{d\tilde{\epsilon}}{dn}, \quad \tilde{\Omega} = \frac{n}{\tilde{\rho}} \frac{d\tilde{\rho}}{dn}, \quad (53)$$

being $\tilde{\epsilon}$ the energy per particle.

3.2 Determining the NP curvature components in 1+3

If we represent the orthonormal tetrad as $\{e_a^\mu\}$, the relation between the NP null tetrad vectors $\{l^\mu, k^\mu, m^\mu, \bar{m}^\mu\}$ and the 1+3 vectors are given by

$$\begin{aligned} l^\mu &= \frac{1}{\sqrt{2}}(e_0^\mu - e_3^\mu) \\ k^\mu &= \frac{1}{\sqrt{2}}(e_0^\mu + e_3^\mu) \\ m^\mu &= \frac{1}{\sqrt{2}}(e_1^\mu - ie_2^\mu) \\ \bar{m}^\mu &= \frac{1}{\sqrt{2}}(e_1^\mu + ie_2^\mu), \end{aligned} \quad (54)$$

and its dual

$$\begin{aligned}
l_\mu &= \frac{1}{\sqrt{2}}(e^3{}_\mu - e^0{}_\mu) \\
k_\mu &= -\frac{1}{\sqrt{2}}(e^0{}_\mu + e^3{}_\mu) \\
m_\mu &= \frac{1}{\sqrt{2}}(e^1{}_\mu - ie^2{}_\mu) \\
\bar{m}_\mu &= \frac{1}{\sqrt{2}}(e^1{}_\mu + ie^2{}_\mu).
\end{aligned} \tag{55}$$

The 4-velocity is equal to $e_0{}^\mu = u^\mu = \frac{1}{\sqrt{2}}(k^\mu + l^\mu)$ and the metric will then be given in terms of this basis by

$$g_{\mu\nu} = 2m_{(\mu}\bar{m}_{\nu)} - 2l_{(\mu}k_{\nu)}. \tag{56}$$

Contracting the Einsteins field equations (20) in coordinate components with the appropriate null vectors and considering the momentum tensor (21) in coordinate components we obtain the relation between the Ricci spinor components and the 1+3 quantities

$$\Phi_{00'} = \Phi_{22'} = \frac{1}{4}(\mu + p + \pi_{33}) \tag{57}$$

$$\Phi_{11'} = \frac{1}{8}(\mu + p + \pi_{11} + \pi_{22} - \pi_{33}) \tag{58}$$

$$\Phi_{01'} = \bar{\Phi}_{10'} = \frac{1}{4}(-\pi_{13} + i\pi_{23}) \tag{59}$$

$$\Phi_{12'} = \bar{\Phi}_{21'} = \frac{1}{4}(\pi_{13} - i\pi_{23}) \tag{60}$$

$$\Phi_{02'} = \bar{\Phi}_{20'} = \frac{1}{4}(\pi_{11} - \pi_{22} - i2\pi_{12}) \tag{61}$$

$$\Lambda = \frac{1}{24}(\mu - 3p). \tag{62}$$

On the lefthand side we have the Ricci spinor components and on the righthand side we have the 1+3 components of the symmetric anisotropic pressure tensor π_{ab} , the energy density scalar μ and isotropic pressure scalar p . If we consider π_{ab} to have zero components then we have a perfect fluid source and the relation between the Ricci spinor components and the 1+3 quantities are those obtained in [30].

Defining the tensor

$$Q_{\mu\nu} = {}^*C_{\mu\nu\rho\sigma}u^\nu u^\sigma = E_{\mu\rho} + iH_{\mu\rho}, \tag{63}$$

the dual part of the Weyl tensor can be expressed in terms of $Q_{\mu\nu}$ as

$$\begin{aligned}
{}^*C_{\mu\nu\rho\sigma}u^\nu u^\sigma &= 8u_{[\mu}Q_{\nu][\sigma}u_{\rho]} + 2g_{\mu[\rho}Q_{\sigma]\nu} - 2g_{\nu[\rho}Q_{\sigma]\mu} \\
&+ 2i\epsilon_{\mu\nu\gamma\delta}u^\gamma u_{[\rho}Q_{\sigma]}^\delta + 2i\epsilon_{\rho\sigma\gamma\delta}u^\gamma u_{[\mu}Q_{\nu]}^\delta.
\end{aligned} \tag{64}$$

Considering that the spinor equivalent of ${}^*C_{\mu\nu\rho\sigma}$ is $\epsilon_{A'B'}\epsilon_{C'D'}\Psi_{ABCD}$ and contracting (64) with the null tetrad vectors one can express the Weyl spinor components in terms of the 1+3 components of $Q_{\mu\nu}$ as

$$\begin{aligned}
\Psi_0 &= \frac{1}{2}(Q_{11} - Q_{22} - 2iQ_{12}) \\
\Psi_1 &= \frac{1}{2}(iQ_{23} - Q_{13}) \\
\Psi_2 &= \frac{1}{2}Q_{33} \\
\Psi_3 &= \frac{1}{2}(iQ_{23} + Q_{13}) \\
\Psi_4 &= \frac{1}{2}(Q_{11} - Q_{22} + 2iQ_{12})
\end{aligned} \tag{65}$$

4 Conformally flat metrics with elastic stress energy tensor

It is well known that the Weyl tensor $C_{\mu\nu\rho\sigma}$ vanishes iff the spacetime is conformally flat. By definition, the metric of a conformally flat spacetime can be written as:

$$ds^2 = F^2(t, x, y, z)(-dt^2 + dx^2 + dy^2 + dz^2). \tag{66}$$

All conformally flat solutions with a perfect fluid, an electromagnetic field, or a pure radiation field are known.

Given that the Ricci spinor components transform under null rotations in the following way

$$\begin{aligned}
\Phi_{00'} &\rightarrow \Phi_{00'} \\
\Phi_{01'} &\rightarrow \Phi_{01'} + \alpha\Phi_{00'} \\
\Phi_{02'} &\rightarrow \Phi_{02'} + 2\alpha\Phi_{01'} + \alpha^2\Phi_{00'} \\
\Phi_{11'} &\rightarrow \Phi_{11'} + \alpha\Phi_{10'} + \bar{\alpha}\Phi_{01'} + \alpha\bar{\alpha}\Phi_{00'} \\
\Phi_{12'} &\rightarrow \Phi_{12'} + 2\alpha\Phi_{11'} + \alpha^2\Phi_{10'} + \bar{\alpha}\Phi_{02'} + 2\alpha\bar{\alpha}\Phi_{01'} + \alpha^2\bar{\alpha}\Phi_{00'} \\
\Phi_{22'} &\rightarrow \Phi_{22'} + 2\alpha\Phi_{21'} + 2\bar{\alpha}\Phi_{12'} + 4\alpha\bar{\alpha}\Phi_{11'} + 2\alpha\bar{\alpha}^2\Phi_{01'} \\
&+ 2\alpha^2\bar{\alpha}\Phi_{10'} + \alpha^2\Phi_{20'} + \bar{\alpha}^2\Phi_{02'} + \bar{\alpha}^2\alpha^2\Phi_{00'},
\end{aligned} \tag{67}$$

with α a complex function, if π_{13} , π_{23} , $\pi_{12} \neq 0$ and $\pi_{13}^2 = \frac{1}{3}(\mu + p + \pi_{33})(\pi_{11} - 2\pi_{22} - \mu - p)$, $\pi_{23}^2 = \frac{1}{3}(\mu + p + \pi_{33})(-2\pi_{11} + \pi_{22} - \mu - p)$, $\pi_{12}^2 = \frac{1}{9}(\pi_{11} - 2\pi_{22} -$

$\mu - p)(-\pi_{11} + 2\pi_{22} - \mu - p)$, one can choose a basis ($\alpha = -\frac{\Phi_{01'}}{\Phi_{00'}}$) such that in that basis $\pi_{13}, \pi_{23}, \pi_{12} = 0$.

Here we study the special case of conformally flat spacetimes with elastic energy-momentum tensor (50), with an anisotropic traceless pressure tensor $\pi_{\mu\nu}$ that can be put in the form

$$\pi_{11} = \pi_{22} = -\frac{1}{2}\pi_{33}, \quad (68)$$

with all other components being null and with zero curvature scalar

$$\Lambda = \frac{1}{24}(\mu - 3p) = 0, \quad (69)$$

and therefore the only non-zero curvature terms are

$$\Phi_{00'} = \frac{1}{2}(2p - \pi_{11}) = \Phi_{22'} \quad (70)$$

$$\Phi_{11'} = \frac{1}{2}(p + \pi_{11}). \quad (71)$$

If we calculate the Bianchi identities for this special case we obtain the following equations for the 1+3 directional derivatives of p and π as well as conditions involving $p, \pi_{11}, n_{\alpha\beta}, a_\alpha, \dot{u}_\alpha, \omega_\alpha, \sigma_{\alpha\beta}, \Omega_\alpha, \Theta$

$$e_0(p) = -\frac{4}{3}p\Theta - 2\sigma_{11}\pi_{11} \quad (72)$$

$$e_0(\pi_{11}) = -4\sigma_{11}p + (\sigma_{11} - \frac{1}{3}\Theta)\pi_{11} \quad (73)$$

$$e_1(p) = -(a_1 - n_{23})\pi_{11} \quad (74)$$

$$e_1(\pi_{11}) = (a_1 - n_{23})\pi_{11} \quad (75)$$

$$e_2(p) = -(a_2 + n_{13})\pi_{11} \quad (76)$$

$$e_2(\pi_{11}) = (a_2 + n_{13})\pi_{11} \quad (77)$$

$$e_3(p) = -\frac{4}{3}\dot{u}_3p + \frac{2}{3}\dot{u}_3\pi_{11} \quad (78)$$

$$e_3(\pi_{11}) = \frac{4}{3}\dot{u}_3p + (3a_3 - \frac{2}{3}\dot{u}_3)\pi_{11} \quad (79)$$

$$4\dot{u}_1p + (\dot{u}_1 - 3a_1 + 3n_{23})\pi_{11} = 0 \quad (80)$$

$$4\dot{u}_2p + (\dot{u}_2 - 3a_2 - 3n_{13})\pi_{11} = 0 \quad (81)$$

$$-2\sigma_{13}p + \frac{1}{4}(3\omega_2 - 6\Omega_2 + \sigma_{13})\pi_{11} = 0 \quad (82)$$

$$-2\sigma_{23}p - \frac{1}{4}(3\omega_1 - 6\Omega_1 - \sigma_{23})\pi_{11} = 0 \quad (83)$$

$$-4\omega_1p + \frac{1}{2}(\omega_1 - 3\sigma_{23})\pi_{11} = 0 \quad (84)$$

$$-4\omega_2 p + \frac{1}{2}(\omega_2 + 3\sigma_{13})\pi_{11} = 0 \quad (85)$$

$$\omega_3 = n_{33} = n_{12} = \sigma_{12} = 0 \quad (86)$$

$$\sigma_{11} = \sigma_{22} = -\frac{1}{2}\sigma_{33} \quad (87)$$

$$n_{11} = n_{22}. \quad (88)$$

We take $\sigma_{13} = \sigma_{23} = \omega_1 = \omega_2 = \omega_3 = \Omega_1 = \Omega_2 = \dot{u}_1 = \dot{u}_2 =$ and $n_{23} = a_1$; $n_{13} = -a_2$ so that (80)-(85) are automatically satisfied and the Bianchi identities reduce to

$$e_0(p) = -\frac{4}{3}p\Theta - 2\sigma_{11}\pi_{11} \quad (89)$$

$$e_0(\pi_{11}) = -4\sigma_{11}p + (\sigma_{11} - \frac{1}{3}\Theta)\pi_{11} \quad (90)$$

$$e_1(p) = 0 \quad (91)$$

$$e_1(\pi_{11}) = 0 \quad (92)$$

$$e_2(p) = 0 \quad (93)$$

$$e_2(\pi_{11}) = 0 \quad (94)$$

$$e_3(p) = -\frac{4}{3}\dot{u}_3 p + \frac{2}{3}\dot{u}_3 \pi_{11} \quad (95)$$

$$e_3(\pi_{11}) = \frac{4}{3}\dot{u}_3 p + (3a_3 - \frac{2}{3}\dot{u}_3)\pi_{11}. \quad (96)$$

As for the Ricci and Einstein field equations they are given by

$$e_0(a_3) = -\frac{1}{3}\dot{u}_3\Theta - \frac{1}{3}a_3\Theta - \dot{u}_3\sigma_{11} - a_3\sigma_{11} \quad (97)$$

$$\begin{aligned} e_1(a_1) + e_2(a_2) + e_3(a_3) &= \frac{3}{2}p - \frac{1}{6}\Theta^2 + \frac{3}{2}\sigma_{11}^2 + 2a_1^2 + 2a_2^2 \\ &+ \frac{3}{2}a_3^2 \end{aligned} \quad (98)$$

$$\begin{aligned} e_3(a_3) - e_0(\sigma_{11}) - \frac{1}{3}e_3(\dot{u}_3) &= \Theta\sigma_{11} - \pi_{11} + \frac{1}{3}\dot{u}_3^2 \\ &+ \frac{1}{3}a_3\dot{u}_3 + p - \frac{1}{9}\Theta^2 + \sigma_{11}^2 + a_3^2 \end{aligned} \quad (99)$$

$$e_3(\sigma_{11}) + \frac{1}{3}e_3(\Theta) = 3a_3\sigma_{11} \quad (100)$$

$$e_3(\Omega_3) + e_0(n_{11}) = -\dot{u}_3\Omega_3 + 2\sigma_{11}n_{11} - \frac{1}{3}\Theta n_{11} \quad (101)$$

$$e_0(\Theta) - e_3(\dot{u}_3) = -\frac{1}{3}\Theta^2 - 6\sigma_{11}^2 - 3p + \dot{u}_3^2 - 2a_3\dot{u}_3 \quad (102)$$

$$e_1(n_{11}) - 2e_3(a_2) = 2a_1n_{11} - 2a_3a_2 \quad (103)$$

$$2e_3(a_1) - e_2(n_{11}) = 2a_2n_{11} + 2a_3a_1 \quad (104)$$

$$e_0(a_1) - \frac{1}{2}e_2(\Omega_3) = -\frac{1}{3}a_1\Theta - a_1\sigma_{11} - a_2\Omega_3 \quad (105)$$

$$e_0(a_2) + \frac{1}{2}e_1(\Omega_3) = -\frac{1}{3}a_2\Theta - a_2\sigma_{11} + a_1\Omega_3 \quad (106)$$

$$e_1(\dot{u}_3) = e_2(\dot{u}_3) = 0 \quad (107)$$

$$e_1(a_3) = e_2(a_3) = 0 \quad (108)$$

$$e_1(\sigma_{11}) = e_2(\sigma_{11}) = 0 \quad (109)$$

$$e_1(\Theta) = e_2(\Theta) = 0. \quad (110)$$

In the following section we study the ODE solutions within this particular case.

4.1 Non rotating ODE solutions

We will start by investigating the case where e_0, e_1, e_2 acting on all quantities are zero in which case the previous equations will reduce to ODEs. A simple study of the Bianchi identities shows that e_0, e_1, e_2 acting on p and π_{11} are zero implies that $\Theta = -3\sigma_{11}$ or $\pi_{11} = -4p$. However, if we apply commutator (16) with $\alpha = 3$ to all quantities involved $p, \pi_{11}, \sigma_{11}, n_{11}, a_1, a_2, a_3, \Omega_3, \Theta$ the conclusion is that $\Theta = 6\sigma_{11}$ otherwise all such quantities are constant also (17) with $\alpha = 1$ and $\beta = 2$ implies that $n_{11} = 0$, (17) with $\alpha = 1$ and $\beta = 3$ implies that $a_1 = 0$ and (17) with $\alpha = 2$ and $\beta = 2$ implies that $a_2 = 0$ otherwise all quantities are constant. Therefore two cases arise.

Case A1 $\pi_{11} + 4p = \Theta - 6\sigma_{11} = a_1 = a_2 = n_{11} = 0$

The Jacobi identities and the Einstein equations in summary give the following ODE system

$$\pi_{11} = 12\sigma_{11}^2 - \frac{4}{3}a_3^2 \quad (111)$$

$$p = -3\sigma_{11}^2 + \frac{1}{3}a_3^2 \quad (112)$$

$$\dot{u}_3 = -a_3 \quad (113)$$

$$e_3(\sigma_{11}) = a_3\sigma_{11} \quad (114)$$

$$e_3(a_3) = -9\sigma_{11}^2 + 2a_3^2 \quad (115)$$

$$e_3(\Omega_3) = a_3\Omega_3. \quad (116)$$

If we now choose the coordinates (t, x, y, z) and the 1+3 basis

$$e_0^\mu = u^\mu = (F(z), 0, 0, 0) \quad (117)$$

$$e_1^\mu = (0, F(z), 0, 0) \quad (118)$$

$$e_2^\mu = (0, 0, F(z), 0) \quad (119)$$

$$e_3^\mu = (0, 0, 0, F(z)), \quad (120)$$

its solution for σ_{11} , a_3 , Ω_3 and F in this basis can be written as

$$\sigma_{11} = \sigma_{11}(z) \quad (121)$$

$$a_3 = \pm(A\sigma_{11}^2 + 9)^{\frac{1}{2}}\sigma_{11} \quad (122)$$

$$F = \frac{a_3\sigma_{11}}{\frac{d\sigma_{11}}{dz}} \quad (123)$$

$$\Omega_3 = Be^{\int \frac{a_3}{F} dz} dz, \quad (124)$$

where A and B are constants and σ_{11} is a non constant function of z . If in particular the solution is shearless and expansionless, equation (115) gives the following solution for a_3

$$a_3 = \frac{1}{\int -\frac{2}{F} dz + C}, \quad (125)$$

where C is a constant.

We note that these equations do not allow a perfect fluid solution since $\pi_{11} = -4p = 0$.

Case A2 $\sigma_{11} = \Theta = n_{11} = a_1 = a_2 = 0$

In this case the solution has zero expansion, is shearless and the only extra condition given by the Jacobi identities is in relation to the non zero component of the angular velocity

$$e_3(\Omega_3) = -\dot{u}_3\Omega_3. \quad (126)$$

The Einstein field equations give a relation for the non zero anisotropic pressure component, the isotropic pressure scalar, non zero component of acceleration and the non zero component of the geometric object a_α

$$\pi_{11} = \frac{1}{2}p - \frac{1}{2}a_3^2 + a_3\dot{u}_3, \quad (127)$$

along with the non linear ODE system in p , π_{11} , a_3 , \dot{u}_3

$$e_3(p) = -\dot{u}_3p - \frac{1}{3}\dot{u}_3a_3^2 + \frac{2}{3}a_3\dot{u}_3^2 \quad (128)$$

$$e_3(\dot{u}_3) = 3p - \dot{u}_3^2 + 2a_3\dot{u}_3 \quad (129)$$

$$e_3(a_3) = \frac{3}{2}p + \frac{3}{2}a_3^2. \quad (130)$$

We note that if $a_3 = 0$ then the equations above give $p = \pi_{11} = 0$ and therefore this solution reduces to the case of a dust solution and the only ODE

one must solve relates to \dot{u}_3 . In the case of zero acceleration $\dot{u}_3 = 0$ then the isotropic pressure p is zero and the system reduces to the ODE in a_3 . Also if we put π_{11} in the equations above we obtain $p = 0$ so therefore this case does not allow for a perfect fluid.

If we choose to solve this system using the basis (117)-(120) the relations between the Ricci spinor components and the 1+3 quantities give $\pi_{11} = -4p$ since $\Phi_{00'} = -\frac{1}{2}\Phi_{11'}$ so that the equations then give $\dot{u}_3 = -a_3$ or $\dot{u}_3 = \frac{1}{2}a_3$. When $\dot{u}_3 = -a_3$ the solution can be obtained by putting $\sigma_{11} = 0$ in equations (111)-(116) and is given by (125). The solution when $\dot{u}_3 = \frac{1}{2}a_3$ is given by

$$a_3 = \frac{1}{\int -\frac{3}{2F}dz + D}, \quad (131)$$

with D a constant. The solution for Ω_3 is the same as in (124).

5 Future Work

A more complicated problem would be to consider a non-rotating, non accelerated solution with null angular velocity ($\omega_\alpha = \dot{u}_\alpha = \Omega_\alpha = 0$), purely spatial components of the commutator functions given by

$$n_{\alpha\beta} = 0 \quad (132)$$

$$a_\alpha = (0, 0, a_3), \quad (133)$$

and diagonal shear

$$\sigma_{\alpha\beta} = \begin{pmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{11} & 0 \\ 0 & 0 & -2\sigma_{11} \end{pmatrix}. \quad (134)$$

The Bianchi identities in this case become

$$e_0(p) = -\frac{4}{3}p\Theta - 2\sigma_{11}\pi_{11} \quad (135)$$

$$e_0(\pi_{11}) = -4\sigma_{11}p + (\sigma_{11} - \frac{1}{3}\Theta)\pi_{11} \quad (136)$$

$$\begin{aligned} e_3(\pi_{11}) &= 3a_3\pi_{11} \\ e_1(p) = e_2(p) = e_3(p) &= 0 \\ e_1(\pi_{11}) = e_2(\pi_{11}) &= 0, \end{aligned} \quad (137)$$

and the Ricci and Einstein field equations

$$e_0(a_3) = -\frac{1}{3}a_3\Theta - a_3\sigma_{11} \quad (138)$$

$$e_3(a_3) = \frac{3}{2}p - \frac{1}{6}\Theta^2 + \frac{3}{2}\sigma_{11}^2 + \frac{3}{2}a_3^2 \quad (139)$$

$$e_0(\sigma_{11}) = \frac{1}{2}p + \pi_{11} + \frac{3}{2}\sigma_{11}^2 + \frac{1}{2}a_3^2 - \frac{1}{18}\Theta^2 - \Theta\sigma_{11} \quad (140)$$

$$e_3(\sigma_{11}) + \frac{1}{3}e_3(\Theta) = 3a_3\sigma_{11} \quad (141)$$

$$e_0(\Theta) = -\frac{1}{3}\Theta^2 - 6\sigma_{11}^2 - 3p \quad (142)$$

$$e_1(a_3) = e_2(a_3) = 0 \quad (143)$$

$$e_1(\sigma_{11}) = e_2(\sigma_{11}) = 0 \quad (144)$$

$$e_1(\Theta) = e_2(\Theta) = 0 \quad (145)$$

$$e_1(\Omega_3) = e_2(\Omega_3) = e_3(\Omega_3) = 0. \quad (146)$$

One must apply the commutators to the quantities p , π_{11} , σ_{11} , Ω_3 , a_3 , Θ . In particular the only commutator not identically satisfied when applied to these quantities is (16) with $\alpha = 3$. This commutator can in fact give rise to further information on their second order covariant derivatives. If we write the previous equations in a basis such as (117)-(120) we see that we are dealing with a non linear PDE system which of course is very difficult to solve so that we should consider generalizing the algorithm described in the ODE case to the study of this more general case.

In the study of non conformally flat sapcetimes with elastic stress-energy tensor it would be very useful to have a 1+3 version of the Karlhede classification of metrics since there are several techniques for obtaining exact solutions of Einsteins equations involving tetrad formulation of these equations in which the Karlhede classification is a useful instrument.

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